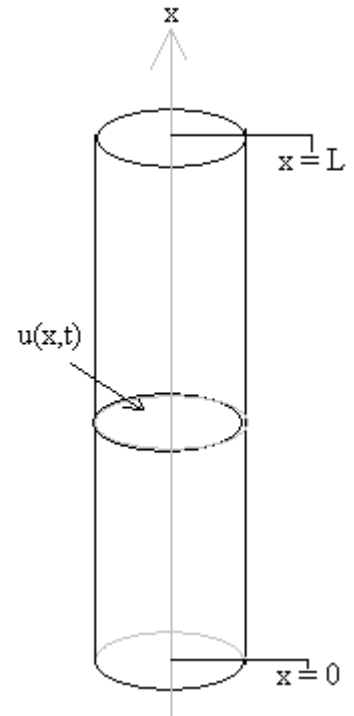


Section II. Derivation of the Heat Conduction Equation for boundary conditions where the ends = zero.

In the first section on MTM derivation, we artificially fixed the problem so that the shelf quit adding heat to the ice instantly as the chamber was isolated by closing the valve. Because the shelf is already warm and there is no way to instantly remove that heat, the problem must be modified to account for the continued warming of the ice by the shelf throughout the time period of the pressure change measurement.

Fortunately, "the conduction of heat in a solid body" is one of the classic partial differential equations of mathematical physics and for simple cases, it has been solved. Consider the ice as shown in the figure to have a distance axis, x , going from bottom to top. $x = 0$ and $x = L$ are the two ends. We will also consider cross sections and for any cross section the temperature is uniform across the area of the section. So $u(x,t)$ is a function only of the distance along the axis and time. That is, at various times, t , the temperature for some given cross section will vary with time. Likewise, for different cross sections and some fixed time, the temperature along the length of the ice will be different.

The differential equation that describes this problem is derived elsewhere [LINK], and is given as EQ#1.



$$\alpha^2 \cdot u_{xx} = u_t \quad 0 < x < L \quad t > 0 \quad \text{EQ\#1}$$

α is called the **thermal diffusivity** and is a constant. The parameter α^2 is defined as follows.

$$\alpha^2 = \frac{\kappa}{\rho \cdot \sigma} \quad \text{EQ\#2}$$

where κ is thermal conductivity, ρ is density and σ is the specific heat of material, in this case, ice. For ice,

$$\rho = 920 \frac{\text{kg}}{\text{m}^3} \quad \kappa = 2.3 \frac{\text{W}}{\text{m} \cdot \text{K}} \quad \sigma = 2093 \cdot \frac{\text{J}}{\text{kg} \cdot \text{K}}$$

The units for α^2 are m^2/s

$$\alpha^2 = 1.194 \cdot 10^{-6} \cdot \frac{\text{m}^2}{\text{s}} \quad \text{tables often give the value in } \text{cm}^2/\text{s} = 0.012 \text{ cm}^2/\text{s}$$

Next, we must assume or measure an initial temperature distribution of the ice and express it as a function of distance, x , at time, $t=0$.

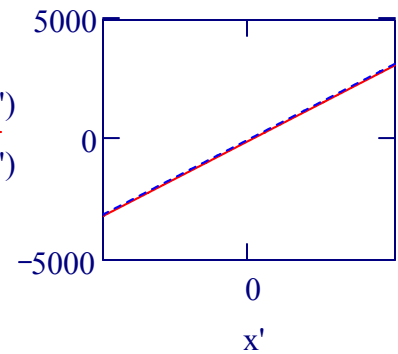
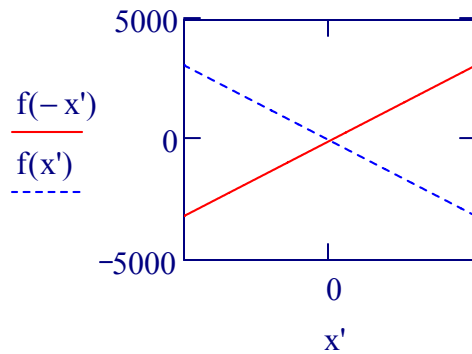
$$u(x, 0) = f(x) \quad 0 \leq x \leq L \quad \text{EQ\#3} \quad \text{where } f(x) \text{ is the initial temperature distribution function. At time zero, a suitable function for the temperature distribution might be given by the following.}$$

Initial conditions

$$v_i := -35 \quad L := .002 \quad v_b := -33$$

In the language of Fourier Transforms, this approximates an 'Odd' function, but on close examination it is neither even nor odd.

$$f(x') := v_b - \frac{v_b - v_i}{L} \cdot x'$$



Finally, we will start the solution by assuming that the ends of the ice are held at fixed temperatures. We know this to be incorrect. Indeed the entire solution is geared toward determining what the temperature at the top of the ice will be whenever the bottom of the ice is held at a constant temperature. Nonetheless, we will first solve for the case where T_b is the fixed temperature at the bottom of the ice and T_i is the fixed temperature at the top of the ice.

Indeed. To start, we will say that $T_b = T_i = 0$. In a later section, I will reduce this to a more general and realistic boundary condition.

$$u(0, t) = 0 \quad u(L, t) = 0 \quad t > 0 \quad \text{Boundary conditions EQ\#4}$$

Equations (1), (3), and (4) are both linear and homogeneous. Thus we can assume that $u(x,t)$ is some function $X(x) \cdot T(t)$

$$u(x, t) = X(x) \cdot T(t) \quad \text{EQ\#5}$$

Substituting EQ#5 into $u(x,t)$ in EQ#1 yields,

$\alpha^2 \cdot X'' \cdot T = X \cdot T'$ EQ#6 where primes refer to ordinary differentiation with respect to the independent variable, whether x or t . Equation 6 is equivalent to

$$\frac{X''}{X} = \frac{1}{\alpha^2} \cdot \frac{T'}{T} \quad \text{EQ\#7} \quad \text{Separation of Variables}$$

In order for EQ#7 to be valid for $0 < x < L, t > 0$, it is necessary that both sides of EQ#7 be equal to the same constant. Otherwise, by keeping one independent variable (say x) fixed and varying the other, one side (the left in this case) of EQ#7 would remain unchanged while the other varied, thus violating the equality. If we call this separation constant "S", then EQ#7 becomes

$$\frac{X''}{X} = \frac{1}{\alpha^2} \cdot \frac{T'}{T} = S \quad \text{EQ\#8}$$

Hence we obtain the following two ordinary differential equations for $X(x)$ and $T(t)$:

$$X'' - S \cdot X = 0 \quad \text{EQ\#9}$$

$$T' - \alpha^2 \cdot S \cdot T = 0 \quad \text{EQ\#10}$$

The partial differential equation of EQ#1 has thus been replaced by two ordinary differential equations. Each of these equations can be readily solved for any value of the separation constant S. The product of two solutions of EQ#9 and EQ#10, respectively, for any value of S provides a solution of the partial differential equation, EQ#1. However, we are interested only in those solutions of EQ#1 also satisfying the boundary conditions, EQ#4. As we will now show, this severely restricts the possible values of S.

Substituting for $u(x,t)$ from EQ#5 in the boundary condition at $x=0$, we obtain

$$u(0, t) = X(0) \cdot T(t) = 0 \quad \text{EQ\#11}$$

If EQ#11 is satisfied by choosing $T(t)$ to be zero for all t , then $u(x,t)$ will be identically zero. That would be unacceptable, since it fails to satisfy the initial condition of EQ#3. Therefore, EQ#11 must be satisfied by requiring that

$$X(0) = 0 \quad \text{EQ\#12}$$

Similarly, the boundary condition at $x = L$ requires that

$$X(L) = 0 \quad \text{EQ\#13}$$

We now want to consider EQ#9 subject to the boundary conditions of EQ#12 and EQ#13.

Two cases must be distinguished, depending on whether $S=0$ or $S \neq 0$.

If $S = 0$, then the general solution to EQ#9 is

$$X(x) = k_1 \cdot x + k_2 \quad \text{EQ\#14}$$

and, in order to satisfy the second boundary condition, we must have $k_1 = 0$. Hence $X(x)$ is identically zero, and thus $u(x,t)$ is also identically zero. As before, this is unacceptable, and we conclude that S must be nonzero.

If $S \neq 0$, it is convenient to replace it by $-\lambda^2$, where λ is a new parameter, not necessarily real. Then EQ#9 becomes

$$X'' + \lambda^2 \cdot X = 0 \quad \text{EQ\#15}$$

and its general solution is

$$X(x) = k_1 \cdot e^{i\lambda \cdot x} + k_2 \cdot e^{-i\lambda \cdot x} \quad \text{EQ\#16}$$

Applying the boundary conditions EQ#12 and EQ#13, we obtain

$$k_1 + k_2 = 0$$

$$k_1 \cdot e^{i\lambda \cdot L} + k_2 \cdot e^{-i\lambda \cdot L} = 0 \quad \text{EQ\#17}$$

Nontrivial solutions of EQ#17 exist iff the determinant of coefficients is zero; that is,

$$\left| \begin{pmatrix} 1 & 1 \\ e^{i\lambda \cdot L} & e^{-i\lambda \cdot L} \end{pmatrix} \right| = e^{(-i)\lambda \cdot L} - e^{i\lambda \cdot L} = 0 \quad \text{EQ\#18}$$

If we now write $\lambda = \mu + iv$, where μ and v are real, then EQ#18 reduces to

$$e^{-i\mu \cdot L} \cdot e^{v \cdot L} - e^{i\mu \cdot L} e^{-v \cdot L} = 0 \quad \text{EQ\#19}$$

Using Euler's relation one obtains

$$e^{i\mu \cdot L} = \cos(\mu \cdot L) + i \cdot \sin(\mu \cdot L) \quad \text{EQ\#20}$$

and separating equation EQ#19 into real and imaginary parts, we find that

$$\left(e^{v \cdot L} - e^{-v \cdot L}\right) \cdot \cos(\mu \cdot L) - i\left(e^{v \cdot L} + e^{-v \cdot L}\right) \cdot \sin(\mu \cdot L) = 0 \quad \mathbf{EQ\#21}$$

Equation EQ#21 is satisfied only if the real and imaginary parts of the left-hand side are separately zero; thus

$$\left(e^{v \cdot L} - e^{-v \cdot L}\right) \cdot \cos(\mu \cdot L) = 0 \quad \mathbf{EQ\#22a}$$

$$i\left(e^{v \cdot L} + e^{-v \cdot L}\right) \cdot \sin(\mu \cdot L) = 0 \quad \mathbf{EQ\#22b}$$

Since $e^{v \cdot L} + e^{-v \cdot L} > 0$ for all values of v and L , EQ#22b requires that $\sin(\mu \cdot L) = 0$.

Hence μ must be chosen so that $\mu \cdot L$ is a multiple of π , that is, $\mu = \frac{n \cdot \pi}{L}$, with n an integer. For that choice of

μ , $\cos(\mu \cdot L) \neq 0$, and EQ#22a reduces to $e^{v \cdot L} - e^{-v \cdot L} = 2 \cdot \sinh(v \cdot L) = 0$ and thus $v = 0$.

Aside: definition: $\sinh(z) = (1/2)(e^z - e^{-z})$.

Since $v=0$, we cannot also have $\mu=0$; otherwise $S=0$, which we have already rejected. Hence

$$\lambda = \mu = \frac{n \cdot \pi}{L} \quad \text{where } n \text{ is a nonzero integer.} \quad \mathbf{EQ\#23}$$

Returning to EQ#17, we see that $k_2 = -k_1$, and hence from EQ#16

$$X(x) = k_1 \cdot \left(e^{i\mu \cdot x} - e^{-i\mu \cdot x}\right)$$

On again using Euler's relation (see EQ#20) we find that $X(x)$ is proportional to $\sin(\mu x)$.

$$X(x) = k_1 \cdot \left(e^{i\mu \cdot x} - e^{-i\mu \cdot x}\right) = k_1 \cdot 2 \cdot i \cdot \sin(\mu \cdot x) \quad \mathbf{EQ\#24}$$

Euler's Relationship used on EQ#24

Alternatively, from the definition of \sinh

$$e^{i\mu \cdot x} = \cos(\mu \cdot x) + i \cdot \sin(\mu \cdot x)$$

$$X(x) = k_1 \cdot 2 \cdot \sinh(\mu \cdot x)$$

$$e^{-i\mu \cdot x} = \cos(-\mu \cdot x) + i \cdot \sin(-\mu \cdot x)$$

$$\left(\cos(\mu \cdot x) + i \cdot \sin(\mu \cdot x)\right) - \left(\cos(-\mu \cdot x) + i \cdot \sin(-\mu \cdot x)\right)$$

$$2 \cdot i \cdot \sin(\mu \cdot x)$$

Summarizing our results to this point, we have shown that we can satisfy the boundary conditions (EQ#4) only if the separation constant, S has certain real negative values, given by

$$S = -\lambda^2 = \frac{-n^2 \cdot \pi^2}{L^2} \quad \text{EQ\#25}$$

where n is a non-zero integer. The corresponding functions X are proportional to $\sin\left(\frac{n \cdot \pi \cdot x}{L}\right)$. Upon introducing the values

of S given by EQ#25 into EQ#10, we find that T(t) is proportional to $e^{\frac{-n^2 \cdot \pi^2 \cdot \alpha^2 \cdot t}{L^2}}$. Hence, neglecting arbitrary constants of proportionality, we conclude that the functions

$$u_n(x, t) = e^{\frac{-n^2 \cdot \pi^2 \cdot \alpha^2 \cdot t}{L^2}} \cdot \sin\left(\frac{n \cdot \pi \cdot x}{L}\right), \quad n = 1, 2, 3, \dots \quad \text{EQ\#26}$$

satisfy the boundary conditions as well as the differential equation EQ#1.

It only remains to satisfy the initial condition, EQ#3.

The differential equation, EQ#1, and the boundary conditions (EQ#4) are linear and homogeneous, and they are satisfied by EQ#26. From the principle of superposition we know that any linear combination of the $u_n(x, t)$ also satisfy the differential equation and boundary conditions. Consequently, we assume that

$$u(x, t) = \sum_{n=1}^m (c_n \cdot u_n(x, t)) = \sum_{n=1}^m \left[c_n \cdot \left(e^{\frac{-n^2 \cdot \pi^2 \cdot \alpha^2 \cdot t}{L^2}} \cdot \sin\left(\frac{n \cdot \pi \cdot x}{L}\right) \right) \right] \quad \text{EQ\#27}$$

where the coefficients c_n are yet undetermined and where m is some positive integer. Since $u(x, t)$ as given by EQ#27 satisfies the differential equation EQ#1 and boundary conditions EQ#4 for any choice of c_n , we wish to investigate whether the c_n can be chosen so as to satisfy the initial conditions (EQ#3) as well.

Although instructive, the solution to this point is not very useful for the principle reason that the boundary conditions, that we have used, constrained both boundaries to zero. **Indeed in the ice problem, the bottom boundary is fixed at T and the top boundary is best approximated by saying the top (end) is perfectly insulated.**

Before moving on to the formation and solution with correct boundaries, we need to look at two examples which when taken together show that the coefficients c_n can be found in the general case if we can express $f(x)$ by means of an infinite series of sine terms.

Unless $f(x)$ takes the form given by Example II, it is not possible to satisfy the initial condition (EQ#3) by means of a finite sum of the form of EQ#27. This suggests that we formally extend the principle of superposition to include infinite series; that is, we assume that m can be replaced by ∞ in EQ#27

$$u(x,t) = \sum_{n=1}^{\infty} \left[c_n \cdot \left(e^{\frac{-n^2 \cdot \pi^2 \cdot \alpha^2 \cdot t}{L^2}} \cdot \sin\left(\frac{n \cdot \pi \cdot x}{L}\right) \right) \right] \quad \text{EQ\#32}$$

We have seen that the individual terms in EQ#32 satisfy the partial differential equation EQ#1 and boundary conditions EQ#4 and that any finite sum of such terms does so as well. We will assume that the infinite series of EQ#32 converges and also satisfies EQ#1 & EQ#4. In order to satisfy the initial condition of EQ#3, we must have

$$u(x,0) = \sum_{n=1}^{\infty} \left(c_n \cdot \sin\left(\frac{n \cdot \pi \cdot x}{L}\right) \right) = f(x) \quad \text{EQ\#33}$$

Now let us suppose that it is possible to express $f(x)$ by means of an infinite series of sine terms,

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \cdot \sin\left(\frac{n \cdot \pi \cdot x}{L}\right) \right) \quad \text{EQ\#34}$$

and that we know how to compute the coefficients b_n in the series. Then, just as in Example II, we can satisfy EQ#33 by choosing $c_n = b_n$ for each n . With the coefficients c_n selected, EQ#32 gives the solution of E

