Heat Equation with Non_Homogeneous Boundary Conditions

Now return to the problem where T1 and T2 represent the end points of the ice cylinder. The differential equation has not changed and is given as EQ#1.

$$\alpha^2 \cdot u_{xx} = u_t \qquad 0 < x < L \qquad t > 0 \qquad \textbf{EQ\#1}$$

but the **boundary conditions** are now

$$u(0,t) = T1 \qquad u(L,t) = T2 \qquad t > 0 \qquad \textbf{EQ\#2}$$

and finally, some **initial condition**, which should not be the steady state condition - as that makes the problem boring.

$$u(x,0) = f(x) \qquad 0 < x < L \qquad \textbf{EQ\#3}$$

Recall that the solution for the homogeneous problem (EQ#2 = 0 and 0 rather than T1 and T2) was found to be

$$u(x,t) = \sum_{n=1}^{\infty} \left(b_n \cdot e^{\frac{-n^2 \cdot \pi^2 \cdot \alpha^2 \cdot t}{L^2}} \cdot \sin\left(\frac{n \cdot \pi \cdot x}{L}\right)\right) \qquad \textbf{EQ\#4 Homogeneous Solution}$$

where the coefficients b_n are determined as follows.

$$b_n = \frac{2}{L} \cdot \int_0^L f(x) \cdot \sin\left(\frac{n \cdot \pi \cdot x}{L}\right) dx \qquad \textbf{EQ\#5}$$

The method of solution for the non-homogeneous problem will be to "make it into a homogenous boundary problem"

As time approaches infinity ($t \Rightarrow \infty$), a steady state distribution, ν**(x)** will be achieved. ν**(x)** must satisfy equation 1.

$$\nu''(x) = 0 \qquad 0 < x < L \qquad \textbf{EQ\#6}$$

and ν**(x)** must satisfy the boundary conditions, EQ#2.

$$\nu(0) = T1 \qquad \nu(L) = T2 \qquad \textbf{EQ\#7}$$

An obvious function that works for EQ#6 and EQ#7 is

$$\nu(x) = T1 + \frac{(T2 - T1) \cdot x}{L} \qquad \textbf{EQ\#8}$$

From EQ#8 one can easily see why the initial condition, f(x) should not be the steady state condition. Rather obviously, nothing would happen.

We can now try to solve EQ#1 with boundary and initial conditions EQ#2, EQ#3 by summing the steady state distribution, ν**(x)** and a new transient temperature distribution, **w(x,t)**.

$$u(x,t) = \nu(x) + w(x,t) \qquad \textbf{EQ\#9}$$

Substituting the equation EQ#9 into the original differential, EQ#1, we get

$\alpha^2(v_{xx} + w_{xx}) = v_t + w_t$ and, since $v''(x) = 0$ and v doesn't change with time $(v_t = 0)$,

$$\alpha^2 \cdot w_{xx} = w_t$$ **EQ#10** (new differential equation)

Next, from EQ#9, EQ#2, and EQ#7

$$w(0,t) = u(0,t) - v(0) = T1 - T1 = 0$$
$$w(L,t) = u(L,t) - v(L) = T2 - T2 = 0$$ **EQ#11** (new boundary conditions)

Next, combining EQ#9 and EQ#3,

$$w(x,0) = u(x,0) - v(x) = f(x) - v(x)$$ **EQ#12** (new initial conditions)

again, notice that if f(x) = v(x) this will be boring.
Also notice that the solution to EQ#10, EQ#11, and EQ12 is the homogeneous problem, the solution of which is given in EQ#4. Thus

$$u(x,t) = v(x) + w(x,t) = T1 + \frac{(T2 - T1) \cdot x}{L} + \sum_{n=1}^{\infty} \left(b_n \cdot e^{\frac{-n^2 \cdot \pi^2 \cdot \alpha^2 \cdot t}{L^2}} \cdot \sin\left(\frac{n \cdot \pi \cdot x}{L}\right)\right)$$

and

$$b_n = \frac{2}{L} \cdot \int_0^L f(x) - \left[T1 + \frac{(T2 - T1) \cdot x}{L}\right] \cdot \sin\left(\frac{n \cdot \pi \cdot x}{L}\right) dx$$

b_n is obtained from the theorem of Fourier which is given below without proof or justification. It just is.

Fourier Theorem: Given any function f(x) which is defined between x equal to -H and +H and which is continuous in this interval or has a limited number of finite discontinuities in this interval, it is possible to express this function in terms of sin and cosine functions as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos\left(\frac{n \cdot \pi \cdot x}{H}\right) + b_n \cdot \sin\left(\frac{n \cdot \pi \cdot x}{H}\right)\right)$$

where the coefficients a_n and b_n are given by

$$a_n = \frac{1}{H} \cdot \int_{-H}^{H} f(x) \cdot \cos\left(\frac{n \cdot \pi \cdot x}{H}\right) dx \qquad b_n = \frac{1}{H} \cdot \int_{-H}^{H} f(x) \cdot \sin\left(\frac{n \cdot \pi \cdot x}{H}\right) dx$$

Definition of an Even Function: f(-x) = f(x)   e.g. cosine is an even function

Definition of an Odd Function: f(-x) = -f(x)   e.g. sine is an odd function.

Caution: Function do not have to be even or odd. Some are neither. They are not periodic on the interval.

Even and Odd Corollary: If f is an even function, then

$$\int_{-L}^{L} f(x)\,dx = 2\cdot\int_{0}^{L} f(x)\,dx$$

If f is an odd function

$$\int_{-L}^{L} f(x)\,dx = 0$$

Cosine Series: If f(x) is an even function, then f(x)*cos( ) is "even". "The product of 2 even numbers is an even number". Ergo, The coefficients from the Fourier Theorem must be

$$a_n = \frac{2}{L}\cdot\int_{0}^{L} f(x)\cdot\cos\left(\frac{n\cdot\pi\cdot x}{L}\right)dx \qquad \text{and } b_n = 0$$

Sine Series: If f(x) is an odd function, the f(x)*sin( ) is "odd". Ergo, the coefficients from the Fourier Theorem must be

$$a_n = 0 \qquad\qquad b_n = \frac{2}{L}\cdot\int_{0}^{L} f(x)\cdot\sin\left(\frac{n\cdot\pi\cdot x}{L}\right)dx$$