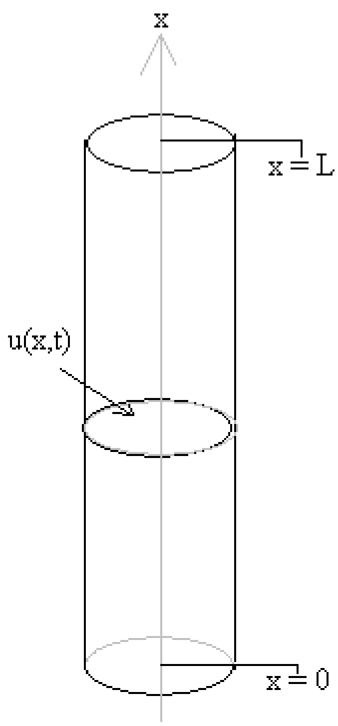
Solution of the Correct Heat Equation for Distribution of Heat through the ICE.

This is a lengthy solution to the heat equation with unusual boundary conditions. In particular, the heat equation is given by

$$\alpha^2 u_{xx} = u_t \qquad 0 < x < L \qquad t > 0 \qquad \textbf{EQ\#1}$$ [Link to Derivation]

u represents temperature of the ice. The subscript xx is meant to represent the second derivitive of temperature along the length of a cylinder of ice. t is time and L is length. **α is thermal diffusivity.**



The two boundaries will be taken as the bottom of the vial, x=0, and the top of the ice, x=L. We will assume that at x=0, there is a constant temperature provided by the shelf and its heaters. The sides and top will be considered to be perfectly insulated.

A consequence of 'perfect insulation' is that temperature can change. When the heat is put into the bottom of the ice and diffuses along and within the ice, if it can't get out, then the temperature will rise as an indication of the added heat. This analogy is imperfect. It fails to account for the loss of mass (and heat) accompanying sublimation which is simultaneously occurring at the upper surface. The analogy works because of the short time period over which we will use the equation. During that period of time ($\leq$ 25 seconds), very little sublimation takes place.

$u(0,t) = T1$ **EQ#2** Boundary condition: Temperature at position x, at all times is constant.

$u'(L,t) = 0$ **EQ#3** Boundary condition: The top is insulated. From the derivation of the heat equation, we know that the instantaneous rate of heat transfer, $H(x_0,t)$, is equal to a constant times the change in temperature with respect to position, $u_x(x_0,t)$. And we are saying that $H(x_0,t)=0$, therefore $u_x(x_0,t)=0$ for position L.

The initial condition to be used is $u(x,0) = f(x)$ **EQ#4** and we will go on to define f(x) as linear in temperature from the bottom to the top of the ice. $f(x) = T1 + \frac{T2 - T1}{L} \cdot x$ **EQ#5** where T1 is the bottom and T2 the top of the ice.

This equation, with these boundary and initial conditions are easily solved numerically. **[See numerical solution]**

I will now show an analytical solution to the problem.

At steady state, $t = \infty$, we can be reasonably assured that T2 = T1. That is, the entire cylinder of ice would come to the shelf temperature, since the problem requires just enough heat input to maintain T1, but no ability for heat to escape. This assumes, and rightly so, that at the initial condition T2 < T1. That is, the top ice interface is colder than the part of the ice in contact with the vial bottom.

I will define a function, $\nu(x)$ = heat distribution at steady state. We have already determined that

$\nu(x) = T1$. **EQ#6**

Note also that $\nu'(x) = 0$ **EQ#7**, (thus it satisfies both boundary conditions), and that $\nu''(x)=0$, thus it also satifies the heat equation. *So if I wanted to wait an infinite amount of time, this is a really easy problem.*

Now I can write u(x,t) as a sum of ν(x) and a different (new) changing temperature distribution.

$$u(x,t) = \nu(x,t) + w(x,t)$$ **EQ#8** remember $\nu(x,t)=T1$

The boundary value problem for w(x,t) is set up by substituting the expression above for u(x,t) into the problem equations [EQ#1,2,3,4].

From EQ#1 $$\alpha^2 \cdot (\nu + w)_{xx} = (\nu + w)_t$$

ergo: $$\alpha^2 \cdot w_{xx} = w_t$$ **EQ#9** since $\nu''(x) = \nu'(t) = 0$

We can establish homogeneous **boundary conditions** for w as follows, defining w(0,t) from EQ#8.

From EQ#8,2 & 6 $$w(0,t) = u(0,t) - \nu(0) = T1 - T1 = 0$$ **EQ#10**

From EQ#8,3 & 7 $$w'(L,t) = u'(L,t) - \nu'(L) = 0$$ **EQ#11**

Finally, combining EQ#8 and the **initial conditio**n, EQ#4,

$$w(x,0) = u(x,0) - \nu(x) = f(x) - \nu(x) = f(x) - T1$$ **EQ#12**

Thus the original problem has been switched around to a homogenous problem in w, with the ultimate solution being EQ#8.

In order to separate the variables, let $w(x,t) = X(x) \cdot T(t)$ **EQ#13**

Substituting EQ#13 into EQ#9 yields

$$\alpha^2 \cdot X'' \cdot T = X \cdot T'$$ **EQ#14**

Separate the variables:

$$\frac{X''}{X} = \frac{1}{\alpha^2} \cdot \frac{T'}{T}$$ **EQ#15**

For equation 15 to be valid, both sides must equal to the same constant. If we were to fix one independent variable and vary the other, then an inequality would result.

$$\frac{X''}{X} = \frac{1}{\alpha^2} \cdot \frac{T'}{T} = \sigma$$ **EQ#16**

and thus we get two equations:

$$X'' - \sigma \cdot X = 0$$ **EQ#17**

$$T' - \sigma \cdot \alpha^2 \cdot T = 0$$ **EQ#18**

Substituting w(x,t) from EQ#13 into the boundary condition at x=0,

$w(0,t) = X(0) \cdot T(t) = 0$ **EQ#19** since T(t) is not always 0, X(0) must = 0.

Thus we have learned that $X(0) = 0$ **EQ#20**

Similarly $X'(L) = 0$ **EQ#21** (boundary condition)

Now I wish to solve equation #17 subject to the boundary conditions of EQ#20 and 21.

Let σ be real. Let $\sigma = -\lambda^2$ where λ is real and positive. Then EQ#17 becomes

$$X'' + \lambda^2 \cdot X = 0$$

which has solution

$$X(x) = k_1 \cdot \sin(\lambda \cdot x) + k_2 \cdot \cos(\lambda \cdot x) \qquad \textbf{EQ\#22}$$

Consider the boundary conditions. First X(0)=0

$$X(0) = k_1 \cdot \sin(\lambda \cdot 0) + k_2 \cdot \cos(\lambda \cdot 0) = 0 \qquad \textbf{EQ\#23}$$

Notice that the second term, the one with a cosine, never evaluates to zero, unless k2 = 0. So, $k_2 = 0$

Notice that the first term will evaluate to zero, thus k1 is not determined.

Now for the 2nd boundary condition. X'(L)=0.

$$X'(L) = \frac{d}{dx}\left(k_1 \cdot \sin(\lambda \cdot L)\right) + \frac{d}{dx}\left(k_2 \cdot \cos(\lambda \cdot 0)\right) = 0$$

$$X'(L) = k_1 \cdot \cos(\lambda \cdot L(x)) \cdot \lambda \cdot \frac{d}{dx}L(x) - k_2 \cdot \sin(\lambda \cdot L(x)) \cdot \lambda \cdot \frac{d}{dx}L(x) = 0$$

The second term above will not be used because we have already determined that k2 = 0.

The first term can only be found as zero when $\lambda = \frac{2 \cdot n - 1}{2 \cdot L} \cdot \pi$ and n= 1,2,3,...∞

In simple terms, n must be an odd number times π.

so $X(x) = k_1 \cdot \sin(\lambda \cdot x) = k_1 \cdot \sin\left[\frac{(2n-1) \cdot \pi \cdot x}{2 \cdot L}\right]$ n= 1,2,3 . . . ∞

Correspondingly, $\sigma = \left[\frac{(2n-1) \cdot \pi}{2 \cdot L}\right]^2$

For these values of σ the solutions to T(t), from EQ#18 are proportional to EXP($-(2n-1)^2\pi^2\alpha^2 t/4L^2$).

Finally, a Fourier series solution can be proposed.

$$w(x,t) = \sum_{n=1}^{\infty} \left[c_n \cdot \left[e^{\frac{-(2n-1)^2 \cdot \pi^2 \cdot \alpha^2 \cdot t}{4 \cdot L^2}}\right] \cdot \sin\left[\frac{(2n-1) \cdot \pi \cdot x}{2 \cdot L}\right]\right]$$

But: Looking at Equation 8

$u(x,t) = v(x) + w(x,t)$ and $v(x) = T1$

so

$$u(x,t) = T1 + \sum_{n=1}^{\infty} \left[c_n \cdot \left[e^{\frac{-(2n-1)^2 \cdot \pi^2 \cdot \alpha^2 \cdot t}{4 \cdot L^2}}\right] \cdot \sin\left[\frac{(2n-1) \cdot \pi \cdot x}{2 \cdot L}\right]\right]$$

To solve for the initial condition, $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} \left[c_n \cdot \sin\left[\frac{(2n-1)\cdot\pi\cdot x}{2\cdot L}\right]\right] = f(x)$$

from Fourier Series,

$$c_n = \frac{2}{L}\cdot\int_0^L (f(x) - T1)\cdot\sin\left[\frac{(2n-1)\cdot\pi\cdot x}{2\cdot L}\right] dx$$

if u(x,0)= f(x) is assumed to be linear at time 0

$$u(x,0) = f(x) = T1 + \frac{T2 - T1}{L}\cdot x$$

and the above integral for c_n can be resolved as follows.

$$c_n = (T1 - T2)\cdot 4\cdot\frac{2\cdot\cos(\pi\cdot n) + 2\cdot\pi\cdot\sin(\pi\cdot n)\cdot n - \pi\cdot\sin(\pi\cdot n)}{\pi^2\cdot(2\cdot n - 1)^2}$$